Self-Consistent Calculation of the Mass and Width of ω from ρ and π

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A self-consistent calculation of the mass and the width of ω following the approximations of Balázs is reported here. The mass and the width found are $m_\omega \approx 650$ MeV and $\Gamma_\omega \approx 1.3$ MeV, respectively, which are to be compared with the experimental values $m_\omega \approx 780$ MeV and $\Gamma_\omega \approx 9.5$ MeV. The D function has a second zero indicating a ghost state at $\sim$820 MeV. This perhaps is due to the neglect of the cut arising from the pion in the crossed channel.

1. INTRODUCTION

THE idea[1] that the necessary force between two particles to create a bound state is generated by the exchange of that bound state itself, has attracted a large number of workers.[2–5] We are interested in considering the ω resonance as the bound state of a ρ and a π. There are various approximation schemes[3,4] to take into account the high-energy behavior of the scattering amplitude. We will follow Balázs[4] in approximating the contributions due to the distant parts of the left-hand cut by two poles. The positions of the poles are fixed as usual[4] by approximating the kernel function by a straight line; the residues at these poles are intermediatory in the self-consistent calculation of the mass of ω and the coupling constant $\gamma_{\omega\rho\pi}$, and hence are determined in the process.

2. POLES AND CUTS IN THE S PLANE

We will treat ρ as a stable particle and neglect the nonnormal threshold, so that according to the Mandelstam representation[6] the scattering amplitude is a function of the complex variables s, t, and u:

$$s=(p_1+q_1)^2,\quad t=(q_1-q_2)^2,\quad u=(p_1-q_2)^2=2m_\rho^2+2m_\pi^2-s-t\,, \qquad (1)$$

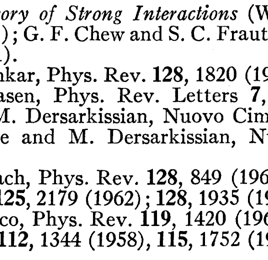

FIG. 1. Diagram for ρ-π scattering.

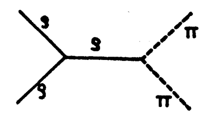

FIG. 2. Input graph for t channel.

where p_1, p_2 are the four-momenta of the incoming and outgoing ρ while q_1, q_2 are those of the π (Fig. 1). The scattering amplitude has then the general form

$$S=1-(\text{const.})\delta^4(p_1+q_1-p_2-q_2)\frac{1}{p_4q_4}M\,, \qquad (2)$$

$$M\sim\frac{g_s(\pi)}{m_\pi^2-s}+\frac{g_s(\omega)}{m_\omega^2-s}+\frac{g_u(\pi)}{m_\pi^2-u}+\frac{g_u(\omega)}{m_\omega^2-u}+\frac{g_t(\rho)}{m_\rho^2-t}+\int_{(m_\rho+m_\pi)^2}^{\infty}ds'\frac{\rho_s(s')}{s'-s}+\int_{(m_\rho+m_\pi)^2}^{\infty}du'\frac{\rho_u(u')}{u'-u}+\int_{4m_\pi^2}^{\infty}dt'\frac{\rho_t(t')}{t'-t}+\text{double integrals.} \qquad (3)$$

The projection of it in the $J=L=1$ channel has in the s plane[6] (i) poles at m_π^2, m_ω^2; cut along the real axis from $(m_\rho+m_\pi)^2$ to ∞; (ii) cuts along the real axis from $(2m_\rho^2+m_\pi^2)$ to $(m_\rho^2-m_\pi^2)^2/m_\pi^2$, from $(m_\rho^2-m_\pi^2)^2/m_\omega^2$ to $2m_\rho^2+2m_\pi^2-m_\omega^2$ and from $-\infty$ to $(m_\rho-m_\pi)^2$; (iii) a cut along the negative real axis $-\infty\leqslant s\leqslant 0$ and along the circumference of the circle $|s|=m_\rho^2-m_\pi^2$. The cut due to the pole in the t channel lies on this circumference at the angles θ where

$$\cos\theta\leqslant\tfrac{1}{2}(m_\rho^2+2m_\pi^2)/(m_\rho^2-m_\pi^2)\,.$$

3. THE FORM OF THE FUNCTIONS $g_s(\omega)$ etc.

Since the spin of the π-ρ system is $0+1=1$, for a given total J, the following four transitions can occur:

$$L\leftrightarrow L\,,\quad L\pm1\leftrightarrow L\pm1\,,\quad\text{and}\quad L-1\leftrightarrow L+1\,.$$

So there must be four independent combinations of the spins describing the scattering. One such choice[7] is

$$a=\epsilon_1\cdot\epsilon_2^*\,,\quad b=(\epsilon_1\cdot P)(\epsilon_2\cdot P)^*\,,\quad c=(\epsilon_1\cdot Q)(\epsilon_2\cdot Q)^*\,,$$

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[4] L. A. P. Balázs, Phys. Rev. **125**, 2179 (1962); **128**, 1935 (1962).
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[7] A. C. Hearn, Nuovo Cimento **21**, 333 (1961).

and

$$d=\tfrac{1}{2}\{(\epsilon_1\cdot P)(\epsilon_2\cdot Q)^*+(\epsilon_1\cdot Q)(\epsilon_2\cdot P)^*\}\,, \tag{4}$$

where ϵ_1 and ϵ_2 are the polarizations of the incoming and outgoing ρ and

$$P=\tfrac{1}{2}(p_1+p_2)\,,\quad Q=\tfrac{1}{2}(q_1+q_2)\,. \tag{5}$$

Any other function that may be imagined to occur in M, i.e., an invariant function symmetric under the exchange

$$(p_1,q_1,\epsilon_1)\leftrightarrow(p_2,q_2,\epsilon_2)\,,$$

can be expressed in terms of the above combinations a, b, c, and d, provided one uses the energy-momentum conservation equation and the equations

$$p_1\cdot\epsilon_1=0=p_2\cdot\epsilon_2\,,$$

which are valid for a spin-one particle.

For example, the contribution of ω in the u channel may be expressed as

$$\begin{aligned}&[\gamma_{\omega\rho\pi}{}^2/(m_\omega{}^2-u)][a\{-\tfrac{1}{2}m_\omega{}^2(2m_\rho{}^2+2m_\pi{}^2-s-m_\omega{}^2)\\&\quad-\tfrac{1}{4}(m_\omega{}^2-(m_\rho+m_\pi)^2)(m_\omega{}^2-(m_\rho-m_\pi)^2)\}\\&\quad+b\{-\tfrac{5}{2}m_\omega{}^2+2m_\rho{}^2-m_\pi{}^2-\tfrac{1}{2}s\}\\&\quad+c(m_\rho{}^2+m_\pi{}^2-\tfrac{1}{2}s-\tfrac{1}{2}m_\omega{}^2)\\&\quad+d(-2m_\rho{}^2+s-m_\omega{}^2)]\,.\end{aligned} \tag{6}$$

4. ANGULAR MOMENTUM PROJECTION

Having $\langle p_2,q_2,\epsilon_2|M|p_1,q_1,\epsilon_1\rangle$ and assuming that ω predominates in the scattering, we want the component corresponding to the ω quantum numbers in the intermediate state. It is convenient to work with the fixed helicity states.[8] We will work in the c.m. system of the incoming ρ and π. Let

$$\langle p_2,q_2,\epsilon_2|M|p_1,q_1,\epsilon_1\rangle=Aa+Bb+Cc+Dd\,. \tag{7}$$

Writing out the ϵ's in terms of the helicities λ and projecting out for various J's, we have

$$\langle p_2,q_2,\epsilon_2|M|p_1,q_1,\epsilon_1\rangle=\frac{1}{4\pi}\sum_J(2J+1)d_{\lambda_1\lambda_2}{}^J(\theta)e^{-i(\lambda_1-\lambda_2)\phi}\langle\lambda_2|M|\lambda_1\rangle\,, \tag{8}$$

so that

$$\langle\lambda_2|M|\lambda_1\rangle=2\pi e^{i(\lambda_1-\lambda_2)\phi}\times\int_0^\pi d(\cos\theta)d_{\lambda_1\lambda_2}{}^J(\theta)\langle p_2,q_2,\epsilon_2|M|p_1,q_1,\epsilon_1\rangle\,. \tag{9}$$

Using[8]

$$\begin{aligned}\langle JMLS|JM\lambda_1\lambda_2\rangle=&[(2L+1)/(2J+1)]^{1/2}\\&\times C(LSJ;0,\lambda_1-\lambda_2)C(s_1s_2s;\lambda_1,-\lambda_2)\,,\end{aligned} \tag{10}$$

we obtain

$$\langle M^{J=L=1}\rangle=\pi\int_0^\pi d(\cos\theta)\times\{-2A\cos\theta+\tfrac{1}{4}(-B-C+D)p^2\sin^2\theta\}\,, \tag{11}$$

where p is the absolute value of the momentum of ρ in c.m. system of the scattering particles:

$$p^2=(1/4s)\{s-(m_\rho+m_\pi)^2\}\{s-(m_\rho-m_\pi)^2\}\,. \tag{12}$$

Using Eqs. (11) and[9]

$$\mathrm{Im}\int_{-1}^{1}\frac{z^n dz}{x-z\pm i\epsilon}=\mp\pi x^n,\quad -1\leq x\leq 1\,, \tag{13}$$

we obtain, for example, the discontinuity of the $J=L=1$ component of Eq. (6) across the ω cut

$$(m_\rho{}^2-m_\pi{}^2)^2/m_\omega{}^2\leqslant s\leqslant 2m_\rho{}^2+2m_\pi{}^2-m_\omega{}^2\,,$$

as

$$\tfrac{1}{2}\pi^2\gamma_{\omega\rho\pi}[x(m_\omega{}^2+s)-x(m_\omega{}^2-s)+\tfrac{1}{2}(1-x^2)(m_\omega{}^2-2m_\rho{}^2+s)]\,, \tag{14}$$

where

$$x=1+\frac{m_\omega{}^2-2m_\rho{}^2-2m_\pi{}^2+s}{m_\rho{}^2+m_\pi{}^2-\frac{1}{2}s-(m_\rho{}^2-m_\pi{}^2)^2/2s}\,. \tag{15}$$

5. CHOICE OF A FUNCTION TO BE WRITTEN AS N/D

From general principles the form of $(p/\sqrt{s})\langle M^{J=L=1}\rangle$ is $e^{i\delta}\sin\delta$ where δ is the phase shift. Moreover, one sees from Eq. (11) that $\langle M^{J=L=1}\rangle$ contains a further p^2, so that the behavior of $(p/\sqrt{s})\langle M^{J=L=1}\rangle$ near $s=0$ is seen to be $(1/s^2)\times(\text{const.})$. To avoid this singular behavior we must multiply $(p/\sqrt{s})\langle M^{J=L=1}\rangle$ by s^2. But then the behavior at very high energies $s\to\infty$ is undesirable. We define therefore, with s_1' and s_2' arbitrary as yet,

$$h(s)=\frac{s^2}{(s-s_1')(s-s_2')}\frac{p}{\sqrt{s}}\langle M^{J=L=1}\rangle\sim(\text{const.})\frac{s^2}{(s-s_1')(s-s_2')}e^{i\delta}\sin\delta\,. \tag{16}$$

This apparently introduces two more kinematical singularities at s_1' and s_2'. However, since we are going to approximate the left-hand cut by two fixed poles as Balázs did, these singularities at s_1' and s_2' will not be extra if they coincide with the Balázs poles. Thus in contradiction to the appearance, no more singularities are added to those existing already in the approximation scheme; only their residues are altered. We write[10]

$$h(s)=N(s)/D(s)\,, \tag{17}$$

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[9] E. Jahnke and F. Emde, *Tables of Functions* (B. G. Teubner, Leipzig, 1938), p. 109.

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where $N(s)$ has only the left cut and $D(s)$ only the right cut. The circular cut and the cut $0\leqslant s\leqslant (m_\rho-m_\pi)^2$ are neglected in the hope that they are weak and their effect is somehow taken into account, at least partially, by the two Balázs poles[4] which we are going to introduce. The cut due to the u-channel π lies above the threshold and is swamped by the unitarity cut. With no further apologies we will neglect it.

The dispersion relations[10] then give

$$D(s)=1-\frac{1}{\pi}(s-s_0)\int_{(m_\rho+m_\pi)^2}^{\infty} ds'\frac{\mathrm{Im}D(s')}{(s'-s)(s'-s_0)}$$
$$=1-\frac{1}{\pi}(s-s_0)\int_{(m_\rho+m_\pi)^2}^{\infty} ds'\frac{N(s')}{(s'-s)(s'-s_0)}\times\mathrm{Im}\{h(s')\}^{-1}, \quad (18)$$

and

$$N(s)=N(s_0)-\frac{1}{\pi}(s-s_0)\int_{-\infty}^{0} ds'\frac{\mathrm{Im}N(s')}{(s'-s)(s'-s_0)}$$
$$-\frac{1}{\pi}(s-s_0)\int_{\omega\ \mathrm{cut}} ds'\frac{D(s')\ \mathrm{Im}h(s')}{(s'-s)(s'-s_0)}. \quad (19)$$

The third term on the right-hand side of Eq. (19) can be estimated explicitly in terms of $D(s_3)$ since $\mathrm{Im}h(s')$ is known from Eqs. (14) and (15), and s_3 is a suitable point on the short ω cut. In the second term there we put $s'-s_0=-(1/x)$ and approximate the resulting kernel by a straight line, obtaining as usual the two-pole approximation. Thus we write

$$N(s)\cong(s-s_0)\sum_{i=0}^{3}\frac{\alpha_i}{s_i-s}, \quad (20)$$

where

$$\alpha_3\cong-\frac{1}{\pi}\frac{D(s_3)}{s_3-s_0}\int_{\omega\ \mathrm{cut}}\mathrm{Im}h(s')ds', \quad (21)$$

$\alpha_0=-N(s_0)$ and α_1, α_2 are unknowns. For convenience we will let s_1', s_2' coincide with s_1, s_2, respectively, and choose the matching point s_0 at $(m_\rho-m_\pi)^2$.

Using Eqs. (20) and (16) we can write Eq. (18) as

$$D(s)=1-\frac{1}{\pi}(s-s_0)\int_{(m_\rho+m_\pi)^2}^{\infty} ds'\times\frac{(s'-s_1)(s'-s_2)}{(s'-s)s'^2}\frac{p'}{\sqrt{s'}}\sum_{i=0}^{3}\frac{\alpha_i}{s_i-s'}, \quad (22)$$

where

$$p'=\left\{\frac{1}{4s'}[s'-(m_\rho+m_\pi)^2][s'-(m_\rho-m_\pi)^2]\right\}^{1/2} \quad (23)$$

As usual we will now express everything in terms of α_1

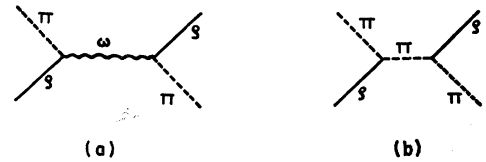

FIG. 3. Input graphs for the identical s and u channels.

and α_2. Near the threshold, δ varies as p^3 or in other words $N(s)$ vanishes at the threshold:

$$0=N[(m_\rho+m_\pi)^2]=-\alpha_0+\sum_{i=1}^{3}\frac{\alpha_i}{s_i-(m_\rho+m_\pi)^2}\times\{(m_\rho+m_\pi)^2-s_0\}. \quad (24)$$

This equation and Eq. (21) express α_0 and α_3 in terms of the unknowns α_1 and α_2 and the known quantities s_0, s_1, and s_2.

To proceed further we need the value of the scattering amplitude and of its first derivative at the matching point. Following Balázs these are estimated by the fixed s dispersion relation and the graphs in Fig. 3:

$$N(s_0)/D(s_0)=0 \quad (25)$$

and

$$(N/D)_{s_0}'=-(8/3)\pi m_\rho m_\pi\gamma_{\omega\rho\pi}^2\times[\{m_\omega^2-(m_\rho-m_\pi)^2\}^{-1}-(m_\rho^2+2m_\pi^2)\times\{6(m_\rho-m_\pi)^2\}^{-1}\{(m_\rho+m_\pi)^2-m_\omega^2\}^{-1}]+\frac{16}{3}\pi\frac{m_\pi}{m_\rho+2m_\pi}\gamma_{\rho\pi\pi}^2. \quad (26)$$

As the π cut has been neglected in the N/D equations, to be consistent we will neglect it here as well, i.e., put $\gamma_{\rho\pi\pi}=0$ in Eq. (26), as no other graph involving the $\rho\pi\pi$ vertex is involved.

Near $s=m_\omega^2$ we have approximately

$$N/D\approx\tfrac{2}{3}\pi\{s-(m_\rho+m_\pi)^2\}\times\{s-(m_\rho-m_\pi)^2\}[\gamma_{\omega\rho\pi}^2/(m_\omega^2-s)], \quad (27)$$

so that

$$D(m_\omega^2)=0, \quad (28)$$

and

$$\gamma_{\omega\rho\pi}^2=\frac{3}{2\pi}\{s-(m_\rho+m_\pi)^2\}^{-1}\{s-(m_\rho-m_\pi)^2\}^{-1}\times[N(m_\omega^2)/-D'(m_\omega^2)]. \quad (29)$$

The formulation of the self-consistent problem is now complete. One starts with some values of m_ω^2 and $\gamma_{\omega\rho\pi}^2$, calculates α_1 and α_2 through Eqs. (20), (22), (25), and (26), and redetermines m_ω^2 and $\gamma_{\omega\rho\pi}^2$ through Eqs. (28) and (29). The input values are varied till the output values agree with them.

6. DISCUSSION

The self-consistent values we find in two iterations are $m_\omega^2 \approx 21.8\ m_\pi^2$ and $\gamma_{\omega\rho\pi}^2 \approx 2.15/m_\pi^2$. The experimental value of m_ω^2 is 31.4 m_π^2. One can estimate the coupling constant $\gamma_{\omega\rho\pi}^2$ from the decay[11] of ω into 3π and the experimental value $\Gamma_\omega \approx 9.5$ MeV.[12] This gives $\gamma_{\omega\rho\pi}^2 \approx 15.4/m_\pi^2$. The agreement is fair enough in high-energy physics.

The curious thing one observes about the computed D function is that it has a second zero at about $S \approx 32.8\ m_\pi^2$ with a positive slope. It may simply be due to the inadequate treatment of the π cut while applying unitarity.

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Bound States and Regge Trajectories in a Vector Meson Exchange Model. I. Application to the $K\bar{K}$ System*

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Partial-wave dispersion relations, extended to noninteger angular momenta are utilized, together with assumptions on the dominance of one-meson exchange to compute the properties of bound states. The exchanged mesons are represented by Regge poles, which lead to a set of equations of generalized Fredholm type when the N/D technique is applied. Bound-state energies in a one-channel system of two spinless particles are computed, as well as the slope of the Regge trajectory which passes through each bound state; the latter is accomplished by an extension of the N/D formalism to angular momenta in the neighborhood of the positive integers. Threshold questions are treated by an approximation for more complex diagrams. The integral equations are solved without further approximations by electronic computer methods. The model is applied to the φ meson as a nearly bound state of the $K\bar{K}$ system in the present work and yields information on S-wave $K\bar{K}$ interactions. Application to future "bootstrap" calculations, the reason for computing the Regge slopes, is discussed, as well as the relationship to the strip approximation.

I. INTRODUCTION

In this paper we study a simple model strong interaction calculation based on the idea that single-vector meson exchange mechanisms are the dominant dynamical singularities in the analytically continued S matrix. A one-channel elastic scattering amplitude for two nonidentical pseudoscalar particles, satisfying a Mandelstam representation, is chosen for definiteness; it is only a matter of detail based on previous analyses to generalize to particles with spin,[1] multichannel reactions,[2] and reactions which lead in this model to complex singularities.[3] We specialize further to discuss the physical problem of the $K\bar{K}$ amplitude, assuming ρ-meson exchange is the dominant interaction. This has physical interest due to the discovery of the φ meson.[4] The hypothesis that the φ is a simple elastic P-wave resonance in the $K\bar{K}$ system is examined; and theoretical reasons are put forth, based on the model calculation, that an isoscalar, scalar meson (σ) should exist. It appears in our model as an S-wave bound state of K and $\bar{K}$.

The main applications of this model, however, are expected to be in "bootstrap" calculations in which the

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